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Block designs with two groups of treatments

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Summary

In this paper procedures of construction of some block designs with two different numbers of replications are discussed. Block designs with two sets of treatments are widely used in practice. The theorems are illustrated by examples.

1. Introduction

Augmented block designs are very useful in research work, particularly in agriculture and medicine, where frequently one or more additional treatments called standard or control treatments are introduced. The literature knows different types of augmented block designs with two groups of treatments, for example reinforced designs introduced by Das (1958), augmented designs by Federer (1956), supplemented designs by Pearce (1960), orthogonally supplemented designs by Caliński (1971).

The main problem here is the augmentation (supplementation) of the basic design with basic treatments by control treatments. This augmentation is realized by the supplementation of additional treatments to the basic design in a definite way.

Therefore, let us consider a block design where v treatments are divided into two groups of v_1 and v_2 treatments ($v = v_1 + v_2$). The treatments of the first group (the standard or basic ones) are replicated r_1 times and the treatments of the second group (the additional or control ones) are replicated r_2 times. The general scheme of such design is:

Key words: augmented block designs, orthogonally supplemented balanced block designs, reinforced block designs, supplemented block designs

d_1
d_2

where $d_1(v_1, b, r_1, \mathbf{k}_1)$ is a block design for the basic treatments, and $d_2(v_2, b, r_2, \mathbf{k}_2)$ is a block design for additional treatments. This design (let us identify it by d) is therefore explicitely described by the incidence matrix of the form

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \end{bmatrix}. \tag{1.1}$$

where $\mathbf{N}_1$, $\mathbf{N}_2$ are incidence matrices of the basic and the additional designs, respectively, and $r_1 v_1 = \mathbf{k}_1'\mathbf{1} = n_1$, $r_2 v_2 = \mathbf{k}_2'\mathbf{1} = n_2$, $n = n_1 + n_2$. The parameters of the d design are: $v = v_1 + v_2$, b, $\mathbf{r} = (r_1\mathbf{1}' \vdots r_2\mathbf{1}')'$, $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$, where b is the number of blocks, $\mathbf{k}$ is the vector of block sizes.

In some cases, it is profitable to add to the d design some additional blocks. Both the basic and the additional treatments are replicated in these blocks in such a way that the requirement of the constant number of replications is preserved for both groups of treatments. The scheme of the design developed in this way has the form

d_1	d_3
d_2	d_4

where d_3 is a design with parameters $v_1, b_2, r_3, \mathbf{k}_3$, and d_4 is a design with parameters $v_2, b_2, r_4, \mathbf{k}_4$.

We analyse in this work only connected designs, i. e. designs where all contrasts of treatment parameters are estimable.

2. Symbols, basic definitions and theorems

The symbols used in this work are the standard ones. Let us remind some of them:

$\mathbf{k}^{\delta} = \mathrm{diag}(k_1, k_2, \ldots, k_b)$

$\mathbf{k}^{-\delta}$ – inverse of matrix $\mathbf{k}^{\delta}$

$\mathbf{M} = \mathbf{r}^{-\delta}\mathbf{N}\mathbf{k}^{-\delta}\mathbf{N}'$

$\mathbf{M}_0 = \mathbf{M} - \mathbf{1}\mathbf{r}'/n$

$\mathbf{\Omega}^{-1} = \mathbf{r}^{\delta} - \mathbf{N}\mathbf{k}^{-\delta}\mathbf{N}' + \mathbf{r}\mathbf{r}'/n$.

Also, the following definitions and theorems are used

Definition 2.1. A block design is said to be efficiency-balanced (EB) design if the matrix $\mathbf{M}_0$ has only two distinct eigenvalues $\mu_0 = 0$ with multiplicity 1 and $\mu \neq 0$ with multiplicity $v - 1$.

Definition 2.2. A block design is said to be partially efficiency-balanced with m efficiency classes (PEB(m)) if

(i) the matrix $\mathbf{M}_0$ has m distinct eigenvalues μ_i (i=1,....,m), whose multiplicity sum is $v - 1$,

(ii) there exist mutually orthogonal idempotent matrices $\mathbf{L}_i$ such that

$$\mathbf{M}_0 = \sum_{i=1}^{m} \mu_i \mathbf{L}_i \ , \quad \sum_{i=1}^{m} \mathbf{L}_i = \mathbf{I} - \mathbf{1}\mathbf{r}'/n \ .$$

Theorem 2.1 (Caliński, 1971). If μ_i (i=1,...,m) are distinct eigenvalues of $\mathbf{M}_0$, then the inverse of matrix $\mathbf{\Omega}^{-1}$ can be written in the form

$$\mathbf{\Omega} \ = [\mathbf{I} + \sum_{i=1}^{m} (\mu_i \,/(1 - \mu_i))\, \mathbf{L}_i \,]\mathbf{r}^{-\delta} \ , \tag{2.1}$$

where $\mathbf{L}_i$ are mutually orthogonal, idempotent matrices corresponding to eigenvalues μ_i so that

$$\mathbf{M}_0 = \sum_{i=1}^{m} \mu_i L_i \ .$$

The third chapter presents the construction of some types of designs with two groups of treatments.

3. Block designs with two groups of treatments of a specific type

3.1. Augmented block designs

Federer (1961) gave the construction, randomization and general methods of analysis of augmented block designs in which additional treatments are replicated less than b times, hence they occur only in a portion of blocks, and the basic design is (a) a randomized complete block design or (b) a lattice design. The d design obtained in this way is a design with incidence matrix (1.1) and matrix $\mathbf{M}_0$ has the form

$$\mathbf{M}_0 \ = \begin{bmatrix} \mathbf{N}_1\mathbf{k}^{-\delta}\mathbf{N}_1'/r_1 - r_1\mathbf{1}\mathbf{1}'\,/n & \mathbf{N}_1\mathbf{k}^{-\delta}\mathbf{N}_2'/r_1 - r_2\mathbf{1}\mathbf{1}'\,/n \\ \mathbf{N}_2\mathbf{k}^{-\delta}\mathbf{N}_1'/r_2 - r_1\mathbf{1}\mathbf{1}'\,/n & \mathbf{N}_2\mathbf{k}^{-\delta}\mathbf{N}_2'/r_2 - r_2\mathbf{1}\mathbf{1}'\,/n \end{bmatrix} . \tag{3.1.1}$$

An example of a type (a) design is the design (columns represent blocks here)

A	A	A	A	A
B	B	B	B	B
C	C	C	C	C
D	D	D	D	D
e	h	k	n	p
f	i	l	o	q
g	j	m	-	-

with parameters $v_1 = 4$, $v_2 = 13$, $b = r_1 = 5$, $r_2 = 1$, $k_1 = k_2 = k_3 = 7$, $k_4 = k_5 = 6$. Capital letters denote the basic treatments and small letters denote additional treatments. The basic treatments are replicated r_1 times and they occur once in each block, and additional treatments occur less than r_1 times (our $r_2 = 1$).

An example of a type (b) design is the design

	Replications								
	1			2			3		
	1	2	3	1	2	3	1	2	3
	A	D	G	A	B	C	A	C	B
	B	E	H	D	E	F	E	D	F
	C	F	I	G	H	I	I	H	G
	j	l	n	p	r	t	u	v	w
	k	m	o	q	s	-	-	-	x
k_{jh}	5	5	5	5	5	4	4	4	5

with parameters $v_1 = 9$, $v_2 = 15$, $b = 9$, $r_1 = 3$, $r_2 = 1$, k_{jh} (j=1,2,3; h=1,2,3) where k_{jh} denotes the size of h-th incomplete block in the j-th complete block.

The augmented triple lattice design is used to illustrate the construction of augmented incomplete block designs. If we add a fourth replication, we get the augmented balanced incomplete block design

	Replication		
		4	
	1	2	3
	A	B	C
	F	D	E
	H	I	G
	y	z	α
	-	-	-
k_{jh}	4	4	4

with parameters $v_1 = 9, v_2 = 18, b = 12, r_1 = 4, r_2 = 1, k_{jh}$ (j=1,2,3,4; h=1,2,3) .

A class of designs, similar to augmented designs analysed by Federer, form the designs presented by Singh and Dey (1979).

Let the basic design be a balanced incomplete block design with the incidence matrix $\mathbf{N}_1$ and parameters v_1, b, r_1, k_1, and let the additional design be a design with the incidence matrix $\mathbf{N}_2 = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Then the incidence matrix of the augmented block design is

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 \\ \mathbf{I} \end{bmatrix}.$$

Theorem 3.1. (Singh and Dey, 1979) The augmented block design with the incidence matrix (1.1), where $\mathbf{N}_2 = \mathbf{I}$, is a partially efficiency-balanced block design with three efficiency classes (PEB(3)) and parameters: $v = v_1+b$, b, $\mathbf{r} = (r_1\mathbf{1}' \ \vdots \ \mathbf{1}')'$, $k = k_1+1$, $\mu_1 = (2r_1-\lambda_1)/r_1(k_1+1)$, $\mu_2 = 1/(k_1+1)$, $\mu_3 = 0$, $\rho_1 = v_1-1$, $\rho_2 = b-v_1$, $\rho_3 = v_1$, where μ_i are the eigenvalues of the matrix $\mathbf{M}_0$ of the design d, ρ_i are the respective multiplicities of μ_i, $\lambda_1 = r_1(k_1-1)/(v_1-1)$. The mutually orthogonal idempotent matrices corresponding to these eigenvalues are:

$$\mathbf{L}_1 = \frac{1}{v_1(r_1-\lambda_1)(2r_1-\lambda_1)} \begin{bmatrix} (r_1-\lambda_1)^2(v_1\mathbf{I}-\mathbf{1}\mathbf{1}') & (r_1-\lambda_1)(v_1\mathbf{N}_1 - k_1\mathbf{1}\mathbf{1}') \\ r_1(r_1-\lambda_1)(v_1\mathbf{N}_1'-k_1\mathbf{1}\mathbf{1}') & r_1(v_1\mathbf{N}_1'\mathbf{N}_1 - k_1^2\mathbf{1}\mathbf{1}') \end{bmatrix},$$

$$\mathbf{L}_2 = \frac{1}{r_1-\lambda_1} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (r_1-\lambda_1)\mathbf{I} + (\lambda_1 k_1/r_1)\,\mathbf{1}\mathbf{1}' - \mathbf{N}_1'\mathbf{N}_1 \end{bmatrix},$$

$$\mathbf{L}_3 = \mathbf{I} - \mathbf{L}_1 - \mathbf{L}_2 - \mathbf{1}\mathbf{r}'/n$$

where $n = b(k_1+1)$. All contrasts within the group of basic treatments have the efficiency factor $e_1 = 1-\mu_1 = \lambda_1 v_1/r_1(k_1+1)$, contrasts within the group of additional treatments have the efficiency factor $e_2 = 1-\mu_2 = k_1/(k_1+1)$, and the contrast efficiency factor between the group of basic treatments and the group of additional treatments equals $e_3 = 1$.

Utilizing the above theorem and formula (2.1) we can determine the form of the matrix $\mathbf{\Omega}$ of an augmented design

$$\mathbf{\Omega} = \begin{bmatrix} \frac{k_1}{\lambda_1 v_1}\mathbf{I} - \frac{r_1-\lambda_1}{\lambda_1 r_1 v_1^2}\mathbf{1}\mathbf{1}' & \frac{1}{\lambda_1 v_1^2}(v_1\mathbf{N}_1 - k_1\mathbf{1}\mathbf{1}') \\ \frac{1}{\lambda_1 v_1^2}(v_1\mathbf{N}_1' - k_1\mathbf{1}\mathbf{1}') & \frac{k_1+1}{k_1}\mathbf{I} + \frac{1}{\lambda_1 k_1 v_1}\mathbf{N}_1'\mathbf{N}_1 - \frac{\lambda_1 v_1 + r_1 k_1}{\lambda_1 r_1 v_1^2}\mathbf{1}\mathbf{1}' \end{bmatrix}.$$

We can calculate the adjusted treatment sum of squares $\mathbf{Q}'\mathbf{\Omega}\mathbf{Q}$, where $\mathbf{Q}$ is the vector of adjusted treatment totals of the form $\mathbf{Q} = (\mathbf{Q}_1' \ \vdots \ \mathbf{Q}_2')'$, while $\mathbf{Q}_1$ and $\mathbf{Q}_2$

are vectors of adjusted treatment totals for the basic and additional treatments respectively

$$\mathbf{Q}'\boldsymbol{\Omega}\mathbf{Q} = \frac{k_1+1}{k_1}\mathbf{Q}_2'\mathbf{Q}_2 + \frac{k_1}{\lambda_1 v_1}\mathbf{Q}_1'\mathbf{Q}_1 + \frac{1}{\lambda_1 v_1 k_1}(\mathbf{N}_1\mathbf{Q}_2)'\,\mathbf{N}_1\mathbf{Q}_2 + \frac{2}{\lambda_1 v_1}\mathbf{Q}_1'\mathbf{N}_1\mathbf{Q}_2 .$$

Having the form of matrix $\boldsymbol{\Omega}$, we can determine the variance of the estimator of comparisons

(i) for two additional treatments

$$\mathrm{Var}(\hat{\tau}_i - \hat{\tau}_{i'}) = \frac{2(\lambda_1 v_1 k_1 + \lambda_1 v_1 + k_1 - \lambda_{ii'})}{\lambda_1 v_1 k_1}\sigma^2 ,$$

where $\lambda_{ii'}$ is the number of basic treatments common between the blocks containing the i-th and i'-th additional treatments,

(ii) for the i-th additional treatment and the j-th basic treatment

$$\mathrm{Var}(\hat{\tau}_i - \hat{\tau}_j) = \begin{cases} \left(\dfrac{(k_1+1)(k_1+\lambda_1 v_1)}{\lambda_1 v_1 k_1} - \dfrac{2}{\lambda_1 v_1}\right)\sigma^2 , \\[2ex] \dfrac{(k_1+1)(k_1+\lambda_1 v_1)}{\lambda_1 v_1 k_1}\sigma^2 , \end{cases}$$

(iii) for two basic treatments

$$\mathrm{Var}(\hat{\tau}_j - \hat{\tau}_{j'}) = \frac{2k_1}{\lambda_1 v_1}\sigma^2 ,$$

where $\hat{\tau}_j$ is the estimate of the effect of the j-th basic treatment, and $\hat{\tau}_i$ is the estimate of the effect of i-th additional treatment ($i \neq i'=1,2,\dots,b;\ j \neq j'=1,\dots,v$).

In particular, if $\mathbf{N}_1 = \mathbf{1}\mathbf{1}'$ and $\mathbf{N}_2 = \mathbf{I}$, i. e. if the basic design is the randomized complete block design, we have

Corollary 3.1.1. The augmented block design with the incidence matrix

$$\mathbf{N} = \begin{bmatrix} \mathbf{1}\mathbf{1}' \\ \mathbf{I} \end{bmatrix}$$

is a partially efficiency-balanced block design with two efficiency classes (PEB(2)) and the parameters: $v = v_1+b$, b, $\mathbf{r} = (b\mathbf{1}' \ \vdots\ \mathbf{1}')'$, $k = v_1+1$, $\mu_1 = 1/(v_1+1)$, $\mu_2 = 0$, $\rho_1 = b-1$, $\rho_2 = v_1$, $e_1 = v_1/(v_1+1)$, $e_2 = 1$,

$$\mathbf{L}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} - \mathbf{1}\mathbf{1}'/b \end{bmatrix}, \quad \mathbf{L}_2 = \mathbf{I} - \mathbf{L}_1 - \mathbf{1}\mathbf{r}'/n ,$$

where $n = b(v_1+1)$.

In this case the matrix $\mathbf{M}_0$ of the augmented design is

$$\mathbf{M}_0 = \frac{1}{k_1+1}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} - \mathbf{1}\mathbf{1}'/b \end{bmatrix},$$

and the adjusted treatment sum of squares is

$$\mathbf{Q}'\Omega\mathbf{Q} = \frac{v_1+1}{v_1}\mathbf{Q}_2'\mathbf{Q}_2 + \frac{1}{b}\mathbf{Q}_1'\mathbf{Q}_1 - \frac{1}{bv_1}(\mathbf{Q}_2'\mathbf{1})^2 .$$

A generalization of the work of Singh and Dey is contained in the papers by Ceranka and Chudzik (1983, 1984).

Corsten (1962) considers the augmented block designs with incidence matrix (1.1). In his work, he assumes that $r_1>r_2$, and that

(i) any two basic treatments occur together in λ_1 blocks,

(ii) each pair of additional treatments occurs together in λ_2 blocks,

(iii) any basic treatment occurs together with any additional treatment in $\lambda_{12} = \lambda_{21}$ blocks.

At the same time, the following conditions are true

$$r_1v_1 + r_2v_2 = n_1 + n_2 = n \quad ,$$

$$r_1(k-1) = \lambda_1(v_1-1) + \lambda_{12}v_2 \quad ,$$

$$r_2(k-1) = \lambda_{12}v_1 + \lambda_2(v_2-1) \quad ,$$

where $k = k_1+k_2$.

Corsten (1962) divides the class of augmented block designs, due to connections between the parameters, into five types A, B, C, D and E.

A generalization of Corsten's work is the work of Ceranka (1975) where the author analyses these designs on the basis of eigenvalues of $\mathbf{M}_0$ and gives the contrast efficiency factors for the particular treatment groups. Let us discuss, for example, the A type design.

The A type design is a design for which $r_2 = \lambda_{12}$, $r_1 = b = \lambda_1$, $r_2>\lambda_2$. The efficiency factor within the group of the basic treatments equals $e_1 = 1-(r_1-\lambda_1)/kr_1 = 1$, the efficiency factor within the group of additional treatments equals $e_2 = 1-(r_2-\lambda_2)/kr_2$, and the contrast efficiency factor between the group of the basic treatments and the group of the additional treatments is $e_3 = 1-(r_1r_2-\lambda_{12}b)/r_1r_2 = 1$.

An example of A type design is the design with the scheme

A	A	A	A	A	A	A
B	B	B	B	B	B	B
c	d	e	f	c	d	c
d	e	f	g	g	h	e
f	g	h	i	h	i	i

and with parameters $v_1 = k_1 = 2$, $b = r_1 = \lambda_1 = 7$, $v_2 = 7$, $r_2 = k_2 = 3$, $\lambda_2 = 1$, $\lambda_{12} = 3$, $\mu_1 = \mu_3 = 0$, $\mu_2 = 2/15$, $\rho_1 = \rho_2 = 1$, $\rho_2 = 6$, $e_1 = e_3 = 1$, $e_2 = 13/15$.

Let us note that the A type design is at the same time an augmented block design analysed by Federer (1961).

3.2 Supplemented block designs

Another type of augmented block designs with two groups of treatments, analysed by Pearce (1960), are designs for which the basic design with constant block sizes is supplemented by one additional treatment occurring in each block the same number of times ($k_2 \geq 1$). The scheme of such a design, using 0 to mark the additional treatment, can be written as

	d_1		
0	…	…	0
.			.
.			.
0	…	…	0

where d_1 is a design with parameters v_1, b, r_1, k_1; the additional design is a design with incidence matrix $\mathbf{N}_2 = \mathbf{k}_2' = k_2\mathbf{1}'$. The obtained design d is a design with parameters $v = v_1{+}1$, b, $\mathbf{r} = (r_1\mathbf{1}' \ \vdots \ bk_2)'$, $k = k_1{+}k_2$. Furthermore, we assume that each pair of basic treatments meets the same (e.g. λ_1) number of times and that each basic treatment meets with the additional treatment the same (e.g. λ_0) number of times

$$\lambda_{12} = \ \dots \ = \lambda_{v-1,v} = \lambda_1 \ ,$$
$$\lambda_{01} = \ \dots \ = \lambda_{0v} = \lambda_0 \ .$$

If $\mathbf{n}_i'$ denotes the i-th row of the incidence matrix $\mathbf{N}$, then we can write

$$\lambda_1 = \mathbf{n}_i'\mathbf{n}_j \ , \quad \text{if} \quad i,j \neq 0, \ i \neq j$$
$$\lambda_0 = \mathbf{n}_0'\mathbf{n}_j$$
$$s_j = \mathbf{n}_j'\mathbf{n}_j \ ,$$

where s_j are diagonal elements of the association matrix $\mathbf{NN}'$. Let us note that $s_j = s$ for each j=1,...,v. It is so because $k\mathbf{r} = k\mathbf{N1} = \mathbf{Nk} = \mathbf{NN}'\mathbf{1}$, hence

$kr_1 = \lambda_0 + s_j + (v_1-1)\lambda_1, j\neq 0$ and $kr_2 = s_0 + v_1\lambda_0$. Therefore the association matrix can be presented in the form

$$\mathbf{NN}' = \begin{bmatrix} (s-\lambda_1)\mathbf{I} + \lambda_1\mathbf{11}' & \lambda_0\mathbf{1} \\ \lambda_0\mathbf{1}' & s_0 \end{bmatrix}. \tag{3.2.1}$$

Designs with constant block sizes and the association matrix (3.2.1) are called by Pearce S type designs. An example of such a design is the design

A	A	A	A
A	B	B	B
B	C	B	C
C	D	C	C
D	D	D	D
0	0	0	0
0	0	0	0

with the incidence matrix and the association matrix being equal respectively to

$$\mathbf{N} = \begin{bmatrix} 2\,1\,1\,1 \\ 1\,1\,2\,1 \\ 1\,1\,1\,2 \\ 1\,2\,1\,1 \\ 2\,2\,2\,2 \end{bmatrix}, \quad \mathbf{NN}' = \begin{bmatrix} 7 & 6 & 6 & 6 & 10 \\ 6 & 7 & 6 & 6 & 10 \\ 6 & 6 & 7 & 6 & 10 \\ 6 & 6 & 6 & 7 & 10 \\ 10 & 10 & 10 & 10 & 16 \end{bmatrix},$$

with parameters $v_1 = b = 4$, $r_1 = 5$, $r_2 = 8$, $k = 7$, $\lambda_0 = 10$, $\lambda_1 = 6$, $s_0 = 16$, $s_1 = 7$.

A special class of supplemented designs form orthogonally supplemented block designs (Caliński, 1971). For this class of designs the basic design is a totally-balanced design and the matrix of the additional design is of the form $\mathbf{N}_2 = \mathbf{r}_2\mathbf{k}'/n$. If there is only one additional treatment and the basic design is a balanced incomplete block design (BIB), then such design is obviously an orthogonally supplemented design and at the same time an S type design. It can be shown that for orthogonally supplemented block designs of type S

$$\mathbf{M}_0 = \begin{bmatrix} (n_1/n)\mathbf{M}_{01} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} (n_1/n)\mathbf{L}_1 & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix},$$

where $\mathbf{M}_{01}$ is the matrix $\mathbf{M}_0$ of the basic design, $\mathbf{L}_1 = \mathbf{I} - \mathbf{11}'/v_1$, and eigenvalues of $\mathbf{M}_{01}$ and $\mathbf{M}_0$ are equal respectively to $\mu_1 = (r_1-\lambda_1)/r_1k_1$, $\mu = (n_1/n)\mu_1 = (r_1-\lambda_1)\ /\ (k_1+1)r_1$, $\rho = v_1-1$.

Furthermore,

$$\mathbf{\Omega} = \begin{bmatrix} \frac{1}{(1-\mu)r_1}\mathbf{I} - \frac{\mu}{(1-\mu)r_1v_1}\mathbf{11}' & \mathbf{0} \\ \mathbf{0}' & \frac{1}{bk_2} \end{bmatrix},$$

and the adjusted treatment sum of squares is

$$\mathbf{Q}'\mathbf{\Omega}\mathbf{Q} = \frac{1}{(1-\mu)r_1}\mathbf{Q}_1'\mathbf{Q}_1 + \frac{1}{bk_2}Q_2^2 - \frac{\mu}{(1-\mu)r_1v_1}(\mathbf{Q}_1'\mathbf{1})^2 .$$

Caliński (1971) showed that addition of supplementary treatments increases the efficiency factor associated with the basic contrasts of the set of treatments. It is so in the case of an orthogonally supplemented design, because all contrasts within the group of basic treatments have the efficiency factor $e_1 = 1-\mu = 1-(n_1/n)\mu_1 = 1-(n_1/n)(1-e)$, where e is the efficiency factor of the basic design, and the remaining contrasts, i.e. the contrasts within the group of additional treatments and the contrast between the group of basic treatments and the group of additional treatments, have the efficiency factors equal to 1 $(e_2 = e_3 = 1)$.

Nigam and Puri (1982) considered as the basic design the m-associate resolvable (PBIB(m)) design with parameters v_1, b, r_1, k_1, λ_i (i=1,...,m). They grouped the blocks in r_1 sets so that each set was a complete replication. Then they added a supplementary treatment number v+1 to each incomplete block of the first set, a treatment number v+1 to each block of the second set, and so on. The incidence matrix of this design has the form

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 \\ \mathbf{I}_r \otimes \mathbf{1}'_{b/r} \end{bmatrix}.$$

We can see that

$$\mathbf{M}_0 = \begin{bmatrix} \sum_{i=1}^{m} \mu_i \mathbf{L}_{i1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \frac{1}{k_1+1}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} - \mathbf{11}'/r_1 \end{bmatrix},$$

where $\mu_i = k_1\mu_{i1}/(k_1+1)$ (i=1,...,m), $\mu_{m+1} = 1/(k_1+1)$ are eigenvalues of $\mathbf{M}_0$, and μ_{i1} are eigenvalues of $\mathbf{M}_{01}$ of the basic design, and mutually orthogonal idempotent matrices $\mathbf{L}_i$ corresponding to the eigenvalues μ_i are

$$\mathbf{L}_i = \begin{bmatrix} \mathbf{L}_{i1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{L}_{m+1} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} - \mathbf{11}'/r_1 \end{bmatrix},$$

where i=1,...,m.

Supplemented block designs with the basic design being a group divisible partially balanced incomplete block design were analysed by Bogacka, Krzyszkowska et al. (1990) and Brzeskwiniewicz (1992).

3.3. Reinforced block designs

A class of designs similar to augmented designs are reinforced block designs which were considered by Das (1958).

Let $d_1(v_1,b_1,r_1,k_1)$ be a balanced incomplete block design. We add $k_2= 1$ additional treatments, all included in each of the blocks, together with $b_2 \geq 0$ new blocks. We obtain a reinforced design with the scheme

$$\begin{array}{|ccc|ccc|}
\hline
 & & & 1 & \dots & 1 \\
 & d_1 & & \vdots & & \vdots \\
 & & & 1 & \dots & 1 \\
\hline
1 & \dots & & & \dots & 1 \\
\vdots & & & & & \vdots \\
1 & \dots & & & \dots & 1 \\
\hline
\end{array}$$

with parameters $v = v_1+k_2$, $b = b_1 + b_2$, $\mathbf{r} = ((r_1+b_2)\mathbf{1}' \vdots (b_1+b_2)\mathbf{1}')'$, $\mathbf{k} = ((k_1+k_2)\mathbf{1}' \vdots (v_1+k_2)\mathbf{1}')'$. Of course, when $b_2 = 0$, we obtain a supplemented design.

Another class of designs are the balanced block designs with two replications presented by Ceranka and Mejza (1979). Let d_1 be a balanced design with incomplete blocks and parameters v_1, b_1, r_1, k_1 with the incidence matrix $\mathbf{N}_1$, and let d_2 be a balanced design with parameters v_1, b_2, r_2, k_2 and with the incidence matrix $\mathbf{N}_2$. We add to each block of the design d_1 only one additional treatment. We obtain a design with v_1+1 treatments with the incidence matrix

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{1}' & \mathbf{0}' \end{bmatrix},$$

or more generally

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_1 \dots \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_2 \dots \mathbf{N}_2 \\ \mathbf{1}' & \mathbf{1}' \dots \mathbf{1}' & \mathbf{0}' & \mathbf{0}' \dots \mathbf{0}' \end{bmatrix},$$

where design d_1 is repeated p times, and design d_2 is repeated q times. The parameters of this design are $v = v_1$, $b = pb_1+qb_2$, $\mathbf{k} = ((k_1+1)\mathbf{1}' \vdots \mathbf{k}'_2)'$, $\mathbf{r} = ((pr_1+qr_2)\mathbf{1}' \vdots pb_1)'$. The condition of the balancing of this design is the equality

$$pv_1(v_1-1)(r_1-\lambda_1) = q(v_1r_2-b_2)(k_1+1)\ .$$

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Układy bloków z dwiema grupami obiektów

Streszczenie

Celem pracy jest pokazanie niektórych metod konstrukcji układow bloków z dwiema grupami obiektów. W praktyce często powstaje zapotrzebowanie na układy, w których liczba replikacji nie jest jednakowa dla każdego obiektu. Praca podsumowuje pewne wyniki w tej dziedzinie, twierdzenia konstrukcyjne zilustrowane sa przykładami.

Słowa kluczowe: ortogonalnie uzupełnione układy bloków, rozszerzone układy bloków, uzupełnione układy bloków, wzmocnione układy bloków